Review: Chain Rule - 10/28/16

1 Chain Rule

Chain Rule: $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$.

Example 1.0.1 *Let* $h(x) = \sqrt{x^2+1}$. *We can break this up into* $f(u) = \sqrt{u}$ *and* $g(x) = x^2+1$. *Then* $f'(u) = \frac{1}{2}u^{-1/2}$ *and* $g'(x) = 2x$. *Then* $h'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}$.

Example 1.0.2 *Let* $h(x) = e^{x^2-3}$. *Let* $f(u) = e^u$ *and* $g(x) = x^2 - 3$. *Then* $f'(u) = e^u$ *and* $g'(x) = 2x$. *Then* $h'(x) = f'(g(x)) \cdot g'(x) = e^{x^2-3} \cdot 2x$.

Example 1.0.3 *Let* $h(x) = \frac{\sin(\cos(x))}{2x-5}$. *Then for the quotient rule, let* $f(x) = \sin(\cos(x))$ *and* $g(x) = 2x - 5$. *Now to find* $f'(x)$, *we need the chain rule. Let* $z(u) = \sin(u)$ *and* $q(x) = \cos(x)$, *so* $z'(u) = \cos(u)$ *and* $q'(x) = -\sin(x)$. *Then by the chain rule,* $f'(x) = \cos(\cos(x)) \cdot (-\sin(x)$. *We also have* $g'(x) = 2$. *Then by the quotient rule,* $h'(x) = \frac{\cos(\cos(x))\cdot(-\sin(x))\cdot(2x-5)-2(\sin(\cos(x)))}{(2x-5)^2}$.

Example 1.0.4 *Let* $h(x) = \cos(\sqrt{x^2+3})$. *To start we can break it up as* $f(u) = \cos(u)$ *and* $g(x) = \sqrt{x^2+3}$, *so* $f'(u) = -\sin(u)$. *To find the derivative of* $g(x)$, *we are going to need the chain rule again. Let* $z(u) = \sqrt{u}$ *and* $q(x) = x^2 + 3$, *then* $z'(u) = \frac{1}{2\sqrt{u}}$ *and* $q'(x) = 2x$, *so* $g'(x) = \frac{x}{\sqrt{x^2+3}}$. *Now we can use the chain rule for the overall function, so* $h'(x) = -\sin(\sqrt{x^2+3}) \cdot \frac{x}{\sqrt{x^2+3}}$.

Practice Problems

1. Find $\frac{d}{dx}\tan(x^2)$.
2. Find $\frac{d}{dx}\tan^2(x)$.
3. Find $\frac{d}{dx}e^{\frac{x+2}{3x}}$.
4. Find $\frac{d}{dx}e^{\sin(3x)}$.
5. Find $\frac{d}{dx}\sqrt{x^2e^x}$.
6. Find $\frac{d}{dx}(3x^{32} - 17)^{100}$.
7. Find $\frac{d}{dx}\cos(\tan(x))$.
8. Find $\frac{d}{dx}\frac{e^{x+3}}{x}$.

Solutions

1. Let $f(u) = \tan(u)$ and let $g(x) = x^2$, so $\tan(x^2) = (f \circ g)(x)$. Then $\frac{d}{dx}\tan(x^2) = \sec^2(x^2) \cdot 2x$.

2. Let $f(u) = u^2$ and let $g(x) = \tan(x)$, so $\tan^2(x) = (f \circ g)(x)$ (since we can think of $\tan^2(x)$ as $(\tan(x))^2$). Then $\frac{d}{dx}\tan^2(x) = 2\tan(x) \cdot \sec^2(x)$.

3. Let $f(u) = e^u$ and $g(x) = \frac{x+2}{3x}$. Then $f'(u) = e^u$ and to find the derivative of g, we need the quotient rule. Let $z(x) = x + 2$ and $q(x) = 3x$, so $z'(x) = 1$ and $q'(x) = 3$. Then $g'(x) = \frac{3x-3(x+2)}{(3x)^2} = \frac{-6}{9x^2} = \frac{-2}{3x^2}$. Now we can use the chain rule to get $\frac{d}{dx}e^{\frac{x+2}{3x}} = e^{\frac{x+2}{3x}} \cdot \frac{-2}{3x^2}$.

4. Let $f(u) = e^u$ and $g(x) = \sin(3x)$, so $f'(u) = e^u$. We need to use the chain rule to find $g'(x)$. Let $z(u) = \sin(u)$ and $q(x) = 3x$, then $g'(x) = 3\cos(3x)$. Then $\frac{d}{dx}e^{\sin(3x)} = e^{\sin(3x)} \cdot 3\cos(3x)$.

5. Let $f(u) = \sqrt{u}$ and $g(x) = x^2e^x$, so $f'(u) = \frac{1}{2\sqrt{u}}$. We need the product rule to find the derivative of $g(x)$. Let $z(x) = x^2$ and $q(x) = e^x$, so $z'(x) = 2x$ and $q'(x) = e^x$. Then $g'(x) = 2xe^x + x^2e^x$. Then $\frac{d}{dx}\sqrt{x^2e^x} = \frac{1}{2\sqrt{x^2e^x}} \cdot (2xe^x + x^2e^x)$.

6. Let $f(u) = u^{100}$ and $g(x) = 3x^{32} - 17$. Then $f'(u) = 100u^{99}$ and $g'(x) = 96x^{31}$. Then $\frac{d}{dx}(3x^{32} - 17)^{100} = 100(3x^{32} - 17)^{99} \cdot (96x^{31})$.

7. Let $f(u) = \cos(u)$ and $g(x) = \tan(x)$, so $f'(u) = -\sin(u)$ and $g'(x) = \sec^2(x)$. Then $\frac{d}{dx}\cos(\tan(x)) = -\sin(\tan(x)) \cdot \sec^2(x)$.

8. We're going to need the quotient rule for this. Let $f(x) = e^{x+3}$ and $g(x) = x$. To find $f'(x)$, we can use the chain rule: let $z(u) = e^u$ and $q(x) = x + 3$, then $z'(u) = e^u$ and $q'(x) = 1$, so $f'(x) = e^{x+3} \cdot 1$. Then by the quotient rule, $\frac{d}{dx}\frac{e^{x+3}}{x} = \frac{e^{x+3}x - e^{x+3}}{x^2}$.